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Fully residually free pro-p groups ☆

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ABSTRACT

We prove that the pro-p completion of the orientable surface group of genus d is residually free pro-p if d is even or $p = 2$. The case when $p \neq 2$, odd $d > 1$ is still open.

1. Introduction

In this paper we study fully residually free pro-p groups. There are no known to us other papers on residual freeness of pro-p groups though the same property for abstract groups is very well understood. The study of abstract residually free groups started in [2] where some examples of residually free groups were given. Later on in [1] the class of fully residually free abstract groups was discussed and shown that it coincides with the class of residually free abstract groups that are transitive commutative. An abstract group is residually free if it embeds in a finite direct product of abstract fully residually free groups [4, Corollary 19]. The study of fully residually free abstract groups was continued by O. Kharlampovich, A. Myasnikov, V. Remeslennikov and Z. Sela and finally lead to the classification of fully residually free groups, nowadays commonly refered to as limit groups, and to the answer of the Tarsky problem by O. Kharlampovich and A. Myasnikov [9] and independently by Z. Sela [12].

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In [6] a special class $\mathcal{L}$ of pro-p groups was defined. The groups of the class $\mathcal{L}$ can be viewed as analogues of limit groups via extension of centralizers. The approach in [6] is to study finitely generated pro-p subgroups of pro-p groups obtained from free pro-p groups of finite rank by finitely many extensions of centralizers. This approach uses substantially the theory of pro-p groups acting on pro-p trees developed by O. Melnikov, L. Ribes, P. Zalesskii. In the abstract case the Bass–Serre theory of abstract groups acting on abstract trees is an important tool to study homological properties of subdirect products of limit groups [3]. In [6] it was shown that the pro-p groups of the class $\mathcal{L}$ are residually torsion-free polyprocyclic and the question was asked whether the groups of the class $\mathcal{L}$ are residually free pro-p or residually torsion-free nilpotent pro-p.

One of the main results of [2] is that for a free abstract group F of finite rank and a cyclic subgroup C of F that is selfcentralized the free amalgamated product $F *_C (C \times \mathbb{Z})$ is residually free. The proofs in [2] use the fact that in an abstract group any element is a finite word on the generators and their inverses, a fact that does not hold in the pro-p case. Here we show that the same result as in [2] holds in the pro-p case under some restriction on the generator of the amalgamated procyclic group $\mathbb{Z}_p$, see Theorem 4.1. This enables us to deduce that the pro-p completion G_d of an orientable surface group of even genus d is residually free pro-p, see Corollary 5.1. In the case $p = 2$ we obtain the same result without restriction on the genus d, see Corollary 5.2. In both cases the considered pro-p surface groups are in the class $\mathcal{L}$ defined in [6].

In the final section we observe that the characterization of fully residually free groups from [1] holds in the pro-p case and since the groups of the class $\mathcal{L}$ are transitive commutative we obtain that G_d is fully residually free pro-p if d is even or $p = 2$, see Theorem 7.3.

We thank Alexei Krassilnikov for his suggestion that help shortening the proof of Lemma 2.1.

2. On lower central series of free abstract and pro-p groups

Denote by $\otimes$ the standard abstract tensor product and if not otherwise stated it is over $\mathbb{Z}$. Let M be an abelian group. Then there is a standard action of the symmetric group S_i on $\bigotimes^i M$ given by permuting the components of the tensor product i.e. for $\alpha \in S_i$ we have $\alpha(m_1 \otimes \cdots \otimes m_i) = m_{\alpha(1)} \otimes \cdots \otimes m_{\alpha(i)}$.

Denote by $\{\gamma_i(H)\}_i$ the lower central series of a group H. In the case of a pro-p group H by definition the terms of the lower central series are closed subgroups.

Lemma 2.1. *Let K be a free abstract group of finite rank with abelianization $M = K/\gamma_2(K)$. Then there is a natural embedding of $\gamma_i(K)/\gamma_{i+1}(K)$ in $\bigotimes^i M$ that sends the class of the left normed commutator $[k_1, \ldots, k_i]$ to $\lambda_i(m_1 \otimes \cdots \otimes m_i)$ for some fixed $\lambda_i \in \mathbb{Z}S_i$ that depends only on i but not on M and every $m_j \in M$ is the image of $k_j \in K$ in M.*

Proof. Consider the free Lie ring L over $\mathbb{Z}$ with $L/[L, L] \simeq M$ and define inductively $\gamma_1(L) = L$, $\gamma_{i+1}(L) = [\gamma_i(L), L]$. Set $L_i = \gamma_i(L)/\gamma_{i+1}(L)$. By [7, Theorem 5.12]

$$\gamma_i(K)/\gamma_{i+1}(K) \simeq L_i, \quad \text{hence } M \simeq L_1.$$

Note that L embeds in the free associative algebra B on n variables $x_1, \ldots, x_n$, where n is the rank of F, so n is the rank of M as a free $\mathbb{Z}$-module. More precisely L is the Lie subring of $B^{(-)}$ generated by $x_1, \ldots, x_n$. Then L_i is isomorphic to the $\mathbb{Z}$-submodule of L spanned by homogeneous Lie monomials of degree i. Any left normed Lie commutator $[l_1, l_2, \ldots, l_i]$, where $l_1, \ldots, l_i \in X = \{x_1, \ldots, x_n\}$ can be written as a linear combination with coefficients 1 or -1 of 2^{i-1} monomials $l_{\pi(1)} \ldots l_{\pi(i)}$ for some $\pi \in S_i$, so

$$[l_1, l_2, \ldots, l_i] = \lambda_i(l_1 \ldots l_i)$$

for some $\lambda_i \in \mathbb{Z}S_i$ that depends only on i. This gives the embeding of $L_i \simeq \gamma_i(K)/\gamma_{i+1}(K)$ in $\bigotimes^i M$ that sends the left normed commutator $[k_1, \ldots, k_i]$ to $\lambda_i(m_1 \otimes \cdots \otimes m_i)$ where $m_j \in M$ is the image of $k_j \in K$ in M for $1 \leqslant j \leqslant i$. $\square$

Denote by $\widehat{\otimes}$ the completed tensor product. If not stated otherwise it is over $\mathbb{Z}_p$.

Corollary 2.2. *Let F be a free pro-p group of finite rank with abelianization $A = F/\gamma_2(F)$. Then there is a natural embedding of $\gamma_i(F)/\gamma_{i+1}(F)$ in $\widehat{\bigotimes}^i A$ sending the class of the left normed commutator $[f_1, \dots, f_i]$ to $\lambda_i(a_1 \widehat{\otimes} \cdots \widehat{\otimes} a_i)$ for some fixed $\lambda_i \in \mathbb{Z}S_i$ that depends only on i but not on A, where every a_j is the image of $f_j \in F$ in A.*

Proof. Let K be a free abstract group such that F is the free pro-p completion of K i.e. K and F have the same basis as free objects. Then $\gamma_i(F)/\gamma_{i+1}(F)$ is a free $\mathbb{Z}_p$-module that is topologically and abstractly finitely generated. Furthermore

$$\gamma_i(F)/\gamma_{i+1}(F) \simeq \big(\gamma_i(K)/\gamma_{i+1}(K)\big) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

In particular

$$A = F/\gamma_2(F) \simeq M \otimes_{\mathbb{Z}} \mathbb{Z}_p, \quad \text{where } M = K/\gamma_2(K).$$

Then

$$\Big(\bigotimes{}^i M\Big) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \bigotimes{}^i_{\mathbb{Z}_p} (M \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \bigotimes{}^i_{\mathbb{Z}_p} A \simeq \widehat{\bigotimes}{}^i_{\mathbb{Z}_p} A, \tag{2.1}$$

where the last isomorphism follows from the fact that A is topologically and hence abstractly finitely generated as $\mathbb{Z}_p$-module. Then the corollary follows directly from Lemma 2.1. $\square$

By going to inverse limit we see that the previous corollary holds if F is a free pro-p group with infinite basis.

Corollary 2.3. *Let F be a free pro-p group with abelianization $A = F/\gamma_2(F)$. Then there is a natural embedding of $\gamma_i(F)/\gamma_{i+1}(F)$ in $\widehat{\bigotimes}^i A$ sending the class of the left normed commutator $[f_1, \dots, f_i]$ to $\lambda_i(a_1 \widehat{\otimes} \cdots \widehat{\otimes} a_i)$ for some fixed $\lambda_i \in \mathbb{Z}S_i$ that depends only on i but not on A, where every a_j is the image of $f_j \in F$ in A.*

3. Fox derivatives and relation modules

We start with two simple lemmas. The simple commutator $[x, y]$ denotes $x^{-1}y^{-1}xy$. For a subset V of a pro-p group we denote by $\langle V \rangle$ the closed subgroup generated by V.

Lemma 3.1. *Let G be a pro-p group and g be an element of G. If $g \in \langle [g, G]^G \rangle$ then $g = 1$.*

Proof. Substituting the inclusion $g \in \langle [g, G]^G \rangle$ in itself i times we get $g \in \langle [g, G, \dots, G]^G \rangle$ where the commutator has length $i + 1$. Thus $g \in \bigcap_i \gamma_i(G) = 1$, where $\{\gamma_i(G)\}_i$ is the lower central series of G. $\square$

Lemma 3.2. *Let F be a free pro-p group with a basis $x_1, \dots, x_m$ and y be an element of F such that $y = [x_1, x_2]t$ and t belongs to the pro-p subgroup F_{m-1} of F generated by $x_2, \dots, x_m$. Then the image of x_2 in $G = F/\langle y \rangle^F$ either has infinite order or is trivial.*

Proof. Assume that the image of x_2 has finite order in $G = F/\langle y \rangle^F$. Let S be the quotient of G by the normal closed subgroup generated by $[x_2, x_1], [x_2, x_3], \dots, [x_2, x_m]$ and let $K = \langle x_2 \rangle \times F(x_1, x_3, \dots, x_m)$, where $F(x_1, x_3, \dots, x_m)$ is a free pro-p group with a basis $x_1, x_3, \dots, x_m$ and $\langle x_2 \rangle$ is infinite procyclic. Then S is the quotient of K by the normal closed subgroup generated by the image t_K of t in K.

Since the image of x_2 has finite order in S we deduce that $t_K \in \langle x_2 \rangle$ in K, hence

$$t \in \langle x_2 \rangle \big\langle [x_2, F]^F \big\rangle \cap F_{m-1} = \langle x_2 \rangle \big\langle [x_2, F_{m-1}]^{F_{m-1}} \big\rangle. \tag{3.1}$$

Put $T = \langle x_2^p \rangle [F_{m-1}, F_{m-1}]$.

Suppose first that $t \in T$. Let B be the quotient of G by the normal closed subgroup generated by the images of $x_3, \dots, x_m$. Then B has two generators x_1 and x_2 and one relation $[x_1, x_2]\lambda_t$ where λ_t is the image of t in B. Since $t \in T$ we have $\lambda_t \in \langle x_2^p \rangle$. Then B is $\mathbb{Z}_p$-by-$\mathbb{Z}_p$ and x_2 has an infinite image in B, a contradiction.

Suppose now that $t \notin T$. Then by (3.1)

$$x_2 \in \langle t \rangle \big\langle [x_2, F_{m-1}]^{F_{m-1}} \big\rangle. \tag{3.2}$$

We use overlining for the images of elements in G. By (3.2)

$$\bar{x}_2 \in \langle \bar{t} \rangle \big\langle [\bar{x}_2, G]^G \big\rangle = \big\langle [\bar{x}_2, G]^G \big\rangle.$$

Then by Lemma 3.1 $\bar{x}_2 = 1$. $\square$

Lemma 3.3. *Let F be a free pro-p group with a basis $x_1, \dots, x_m$, C be the closed subgroup of F generated by an element y of F such that $y = [x_1, x_2]t$, t belongs to the pro-p subgroup of F generated by $x_2, \dots, x_m$ and the image of x_2 in $F/\langle y \rangle^F$ is non-trivial. Let*

$$\pi_j : F \to K_j = F/\big\langle y^{p^j} \big\rangle^F$$

be the canonical projection and R_j be the abelianization of $Ker(\pi_j)$. Then there is an isomorphism of right $\mathbb{Z}_p[[K_j]]$-modules

$$R_j \simeq \mathbb{Z}_p \,\widehat{\otimes}_{\mathbb{Z}_p[[C_j]]}\, \mathbb{Z}_p[[K_j]],$$

where $C_j = \pi_j(C)$ and the action of K_j on R_j is induced by the action of F on R_j via conjugation.

Proof. Consider the exact sequence

$$0 \to R_j \xrightarrow{\partial_2} \bigoplus\nolimits_{1 \leqslant i \leqslant m} e_i \mathbb{Z}_p[[K_j]] \xrightarrow{\partial_1} \mathbb{Z}_p[[K_j]] \xrightarrow{\partial_0} \mathbb{Z}_p \to 0,$$

where the map ∂_0 is augmentation, $e_i\mathbb{Z}_p[[K_j]]$ is a free cyclic $\mathbb{Z}_p[[K_j]]$-module, $\partial_1(e_i) = \pi_j(x_i) - 1 \in \mathbb{Z}_p[[K_j]]$ and ∂_2 is given by Fox derivatives. Namely, there are free Fox derivatives i.e. continuous maps

$$\frac{\partial}{\partial_{x_i}} : F \to \mathbb{Z}_p[[F]]$$

given by

$$\frac{\partial}{\partial_{x_i}}(f_1 f_2) = \frac{\partial}{\partial_{x_i}}(f_2) + \left(\frac{\partial}{\partial_{x_i}}(f_1)\right) f_2 \quad \text{and} \quad \frac{\partial}{\partial_{x_i}}(x_j) = \delta_{i,j} \quad \text{for } f_1, f_2 \in F,\ 1 \leqslant i, j \leqslant m,$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. We denote by the same letter π_j the extension of π_j to a ring homomorphism $\mathbb{Z}_p[[F]] \to \mathbb{Z}_p[[K_j]]$. Then for $f \in Ker(F \to K_j)$ and for the image r of f in R_j we have

$$\partial_2(r) = \sum_{1\leqslant i\leqslant m} e_i \pi_j\left(\frac{\partial}{\partial_{x_i}}(f)\right).$$

Note that $R_j \simeq \partial_2(R_j)$ is a cyclic $\mathbb{Z}_p[\![K_j]\!]$-module generated by

$$\sum_{1\leqslant i\leqslant m} e_i \pi_j\left(\frac{\partial}{\partial_{x_i}}\big(y^{p^j}\big)\right).$$

Since $y = [x_1, x_2]t$ where t belongs to the pro-p subgroup of F generated by $x_2, \dots, x_m$ we have

$$\pi_j\left(\frac{\partial}{\partial_{x_1}}\big(y^{p^j}\big)\right) = \pi_j\left(\frac{\partial}{\partial_{x_1}}(y)\right) \sum_{0\leqslant s\leqslant p^j-1} \pi_j(y)^s$$

and

$$\pi_j\left(\frac{\partial}{\partial_{x_1}}(y)\right) = \pi_j\left(\frac{\partial}{\partial_{x_1}}\big([x_1, x_2]\big)\right)\pi_j(t) = \big(1 - \pi_j\big(x_2^{x_1}\big)^{-1}\big)\pi_j(x_2)\pi_j(t).$$

Let $\lambda \in ann_{\mathbb{Z}_p[\![K_j]\!]}(R_j)$, so for every $1 \leqslant i \leqslant m$ we have in $\mathbb{Z}_p[\![K_j]\!]$

$$\pi_j\left(\frac{\partial}{\partial_{x_i}}\big(y^{p^j}\big)\right)\lambda = 0.$$

In particular

$$0 = \pi_j\left(\frac{\partial}{\partial_{x_1}}\big(y^{p^j}\big)\right)\lambda = \big(1 - \pi_j\big(x_2^{x_1}\big)^{-1}\big)\pi_j(x_2)\pi_j(t)\left(\sum_{0\leqslant s\leqslant p^j-1} \pi_j(y)^s\right)\lambda. \tag{3.3}$$

By Lemma 3.2 $\pi_0(x_2)$ has infinite order in K_0, hence $\pi_j(x_2)$ has infinite order in K_j, so $\pi_j(x_2^{x_1})$ has infinite order in K_j. Then by [5, Proposition 2] $(1 - \pi_j(x_2^{x_1})^{-1})$ is not a zero divisor of $\mathbb{Z}_p[\![K_j]\!]$, so by (3.3)

$$0 = \left(\sum_{0\leqslant s\leqslant p^j-1} \pi_j(y)^s\right)\lambda.$$

Finally let $M = \mu(K_j/C_j)$ where $\mu : K_j/C_j \to K_j$ is a continuous section of $K_j \to K_j/C_j = \{C_j g\}_{g\in K_j}$ and thus M is a profinite space but not a profinite group. Recall that $C_j = \langle \pi_j(y)\rangle \simeq \mathbb{Z}_p/p^j\mathbb{Z}_p$. Then $\mathbb{Z}_p[\![K_j]\!] = \bigoplus_{0\leqslant i\leqslant p^j-1} \pi_j(y)^i \mathbb{Z}_p[\![M]\!]$. It follows that

$$\lambda = \sum_{0\leqslant i\leqslant p^j-1} \pi_j(y)^i \lambda_i \quad \text{for some } \lambda_i \in \mathbb{Z}_p[\![M]\!].$$

In particular

$$0=\Big(\sum_{0\leqslant s\leqslant p^j-1}\pi_j(y)^s\Big)\lambda=\Big(\sum_{0\leqslant s\leqslant p^j-1}\pi_j(y)^s\Big)\cdot\Big(\sum_{0\leqslant i\leqslant p^j-1}\pi_j(y)^i\lambda_i\Big)$$
$$=\Big(\sum_{0\leqslant i\leqslant p^j-1}\pi_j(y)^i\Big)\cdot\Big(\sum_{0\leqslant i\leqslant p^j-1}\lambda_i\Big),$$

hence $\sum_{0\leqslant i\leqslant p^j-1}\lambda_i=0$ and so

$$\lambda=\sum_{0\leqslant i\leqslant p^j-1}\pi_j(y)^i\lambda_i=\sum_{0\leqslant i\leqslant p^j-1}\big(\pi_j(y)^i-1\big)\lambda_i\in\big(\pi_j(y)-1\big)\mathbb{Z}_p[[K_j]].$$

Thus

$$ann_{\mathbb{Z}_p[[K_j]]}(R_j)\subseteq\big(\pi_j(y)-1\big)\mathbb{Z}_p[[K_j]].$$

Since the converse inclusion is obvious we obtain

$$ann_{\mathbb{Z}_p[[K_j]]}(R_j)=\big(\pi_j(y)-1\big)\mathbb{Z}_p[[K_j]].$$

This completes the proof. □

4. A general embedding

Let $F=F(x_1,\dots,x_m)$ be a free pro-p group of finite rank m with free generators $x_1\dots x_m$, $G=F\coprod_C A$, where $A=C\times M$, $C=\langle y\rangle$ and $M=\langle z\rangle\simeq\mathbb{Z}_p$ such that y is of the form $y=[x_1,x_2]t$, t belongs to the pro-p subgroup of F generated by $x_2,\dots,x_m$.

For every positive integer j fix a copy F_j of F and denote by

$$\pi_j:\prod_{i\geqslant 1}F_i\to F_j$$

the canonical projection.

Theorem 4.1. *Let*

$$\theta:G\to\prod_{j\geqslant 1}F_j$$

be the homomorphisms of pro-p groups such that

$$\pi_j\theta:G\to F_j$$

is the identity map on F and sends z to y^{p^j}. If the image of x_2 in $F/\langle y\rangle^F$ is non-trivial, then θ is injective.

Proof. 1. The map ρ.

Let H be the normal closure of z in G i.e. the smallest closed normal subgroup of G containing z. Then by the pro-p version of the Kurosh Subgroup Theorem [8, Theorem 5.6]

$$H = \coprod_{q \in W} \langle z \rangle^q,$$

where $W = \delta(F/C)$, $\delta : F/C \to F$ is a continuous section of the canonical projection map $F \to F/C$, $F/C = \{Cg\}_{g \in F}$ and so W is a profinite subspace of F but not a profinite subgroup of F. Note that

$$\langle z \rangle^q \simeq \langle z \rangle \simeq \mathbb{Z}_p,$$

so H is a free pro-p group. The map θ restricts to the map

$$\rho = \theta|_H : H \to M_0 := \prod_{j \geqslant 1} \langle y^{p^j} \rangle^{F_j}.$$

2. The map ρ_1.

Let

$$K_j = F_j / \langle y^{p^j} \rangle^{F_j}$$

and R_j be the abelianization of the kernel of the canonical projection

$$F_j \to K_j$$

i.e. R_j is a relation module of K_j. Then by the description of W the map ρ induces a map on the abelianizations of both H and M_0

$$\rho_1 : z\mathbb{Z}_p[[F/C]] \to \prod_{j \geqslant 1} R_j,$$

where $z\mathbb{Z}_p[[F/C]]$ is a free cyclic right pro-p $\mathbb{Z}_p[[F/C]]$-module with a free generator z. As every K_j is 1-related the relation module R_j is a cyclic right pro-p $\mathbb{Z}_p[[K_j]]$-module. We think of R_j as a right pro-p $\mathbb{Z}_p[[F_j]]$-module via the canonical projection $F_j \to K_j$.

By Lemma 3.3 for every $j \geqslant 1$ we have

$$R_j \simeq \mathbb{Z}_p \widehat{\otimes}_{\mathbb{Z}_p[[C_j]]} \mathbb{Z}_p[[K_j]],$$

where C_j is the image of C in K_j via the canonical projection. Then

$$R_j \simeq \mathbb{Z}_p \widehat{\otimes}_{\mathbb{Z}_p[[N_j]]} \mathbb{Z}_p[[F]] \simeq \mathbb{Z}_p[[F/N_j]],$$

where N_j is the closed subgroup of $F_j = F$ generated by C and $\langle y^{p^j} \rangle^F$ and $F/N_j = \{N_j g\}_{g \in F}$ is a profinite space.

3. The map $\rho_{1,j}$.

Define

$$\rho_{1,j} : z\mathbb{Z}_p[[F/C]] \to \mathbb{Z}_p[[F/N_j]],$$

as the map $\theta_j \rho_1$ after identifying $\mathbb{Z}_p[[F/N_j]]$ with R_j, where θ_j is the projection of $\prod R_j$ to R_j.

Observe that for the image $\bar{1}$ of 1 in $\mathbb{Z}_p[[F/N_j]]$ we have

$$\rho_{1,j}(z) = \bar{1}.$$

4. The maps ρ_i and μ_i.

Let

$$\rho_i : \gamma_i(H)/\gamma_{i+1}(H) \to \prod_{j \geqslant 1} \gamma_i\big(\big\langle y^{p^j}\big\rangle^{F_j}\big)/\gamma_{i+1}\big(\big\langle y^{p^j}\big\rangle^{F_j}\big)$$

be the map induced by ρ. By Corollary 2.3 there is $\lambda_i \in \mathbb{Z}S_i$ such that

$$\gamma_i(H)/\gamma_{i+1}(H) \simeq \lambda_i\Big(\widehat{\bigotimes}^i \big(H/[H,H]\big)\Big) = \lambda_i\Big(\widehat{\bigotimes}^i \big(z\mathbb{Z}_p[[F/C]]\big)\Big)$$

and

$$\gamma_i\big(\big\langle y^{p^j}\big\rangle^{F_j}\big)/\gamma_{i+1}\big(\big\langle y^{p^j}\big\rangle^{F_j}\big) \simeq \lambda_i\Big(\widehat{\bigotimes}^i R_j\Big).$$

Then the map

$$\rho_i : \lambda_i\Big(\widehat{\bigotimes}^i \big(z\mathbb{Z}_p[[F/C]]\big)\Big) \to \prod_{j \geqslant 1} \lambda_i\Big(\widehat{\bigotimes}^i R_j\Big)$$

extends to a map

$$\mu_i : \widehat{\bigotimes}^i \big(z\mathbb{Z}_p[[F/C]]\big) \to \prod_{j \geqslant 1} \widehat{\bigotimes}^i R_j.$$

The map μ_i combined with the projection on jth coordinate of the direct product $\prod_{j \geqslant 1} \widehat{\bigotimes}^i R_j$ is the map

$$\mu_{i,j} = \widehat{\bigotimes}^i \rho_{1,j} : \widehat{\bigotimes}^i_{\mathbb{Z}_p} \big(z\mathbb{Z}_p[[F/C]]\big) \to \widehat{\bigotimes}^i R_j = \widehat{\bigotimes}^i \mathbb{Z}_p[[F/N_j]].$$

5. ρ is injective.

Note that C is the intersection $\bigcap_{\alpha \geqslant 1} N_\alpha$ and $\mathbb{Z}_p[[F/C]]$ is the inverse limit of $\mathbb{Z}_p[[F/N_\alpha]]$. Then

$$\bigcap_{\alpha > 0} Ker\Big(\widehat{\bigotimes}^i \mathbb{Z}_p[[F/C]] \to \widehat{\bigotimes}^i \mathbb{Z}_p[[F/N_\alpha]]\Big) = 0$$

and so

$$\bigcap_{j \geqslant 1} Ker(\mu_{i,j}) = 0. \tag{4.1}$$

Thus μ_i and ρ_i are injective for every i and so ρ is injective.

6. Proof that θ is injective.

Let

$$\widetilde{\theta} : F = G/H \to \prod_{j \geqslant 1} F_j / \langle y^{p^j} \rangle^{F_j}$$

be the map induced by θ. Since $\bigcap_{j\geqslant 1} \langle y^{p^j} \rangle^F = 1$ the map $\widetilde{\theta}$ is injective. Together with the fact that ρ is injective this implies that θ is injective as required. □

5. Orientable surface pro-p groups

Corollary 5.1. *Every orientable surface pro-p group*

$$G_d = \langle x_1, \ldots, x_{2d} \mid [x_1, x_2] \ldots [x_{2d-1}, x_{2d}] = 1 \rangle,$$

where d is even, is residually free pro-p. In particular G_d is residually torsion-free nilpotent pro-p.

Proof. The group G_d is a free amalgamated pro-p product of F_1 and F_2, where F_1 is a free pro-p group with a basis $x_1, \ldots, x_d$ and F_2 is a free pro-p group with a basis $x_{d+1}, \ldots, x_{2d}$ i.e.

$$G_d = F_1 \coprod_{[x_1,x_2]\ldots[x_{d-1},x_d] = [x_{2d},x_{2d-1}]\ldots[x_{d+2},x_{d+1}]} F_2.$$

We identify F_1 and F_2 with a free pro-p group F with basis $y_1, \ldots, y_d$ via the identification that sends x_i and x_{2d-i} to y_i. Then

$$G_d \simeq F \coprod_C F^a \subseteq F \coprod_C A,$$

where C is the procyclic subgroup of F generated by $[y_1, y_2] \ldots [y_{d-1}, y_d]$ and $A = C \times \langle a \rangle$, $\langle a \rangle \simeq \mathbb{Z}_p$. By Theorem 4.1 $F \coprod_C A$ is residually free pro-p, hence G_d is residually free pro-p. □

Corollary 5.2. *Every orientable surface pro-2 group*

$$G_d = \langle x_1, \ldots, x_{2d} \mid [x_1, x_2] \ldots [x_{2d-1}, x_{2d}] = 1 \rangle,$$

is residually free pro-2. In particular every G_d is residually torsion-free nilpotent pro-2.

Proof. First by the previous corollary we may assume that d is odd. Note that the trivial $\mathbb{Z}_p[[G_d]]$-module $\mathbb{Z}_p$ has a free resolution

$$0 \to \mathbb{Z}_p[[G_d]] \to \mathbb{Z}_p[[G_d]]^{2d} \to \mathbb{Z}_p[[G_d]] \to \mathbb{Z}_p \to 0,$$

hence the Euler characteristic $\chi(G_d) = 2 - 2d$ depends only on d. On other hand an open subgroup of the orientable pro-p surface group $H = G_{2k} = \langle x_1, \ldots, x_{4k} \mid [x_1, x_2] \ldots [x_{4k-1}, x_{4k}] \rangle$ is again a pro-p group of the same type i.e. a pro-p completion of an abstract orientable surface group. Let M be a subgroup of H of index 2^s. Then $\chi(M) = 2^s \chi(H) = 2^s(2 - 4k) = 2 - 2d$ for $d - 1 = 2^s(2k - 1)$. Then $M \simeq G_d$.

Note that for any odd number d there are unique positive integers s, k such that $d - 1 = 2^s(2k - 1)$. Then by Corollary 5.1 $H = G_{2k}$ is residually free pro-2, hence M is residually free pro-2. This completes the proof. □

6. Free pro-p products

Theorem 6.1. *Let F_0 be a free pro-p group of finite rank and G be the free pro-p product $F_0 \coprod \mathbb{Z}_p^2$. Then G is residually free pro-p. In particular G is residually torsion-free nilpotent pro-p.*

Proof. Let F_2 be a free pro-p group with basis $\{x, y\}$. Then by the universal property of free amalgamated pro-p products we have

$$\Big(F_2 \coprod F_0\Big) \coprod_{\langle y\rangle} \big(\langle y\rangle \times \mathbb{Z}_p\big) \simeq F_0 \coprod \Big(F_2 \coprod_{\langle y\rangle} \big(\langle y\rangle \times \mathbb{Z}_p\big)\Big).$$

By [11] free amalgamated pro-p products over procyclic subgroups are proper, so $\mathbb{Z}_p^2 \simeq \langle y\rangle \times \mathbb{Z}_p$ embeds in $F_2 \coprod_{\langle y\rangle}(\langle y\rangle \times \mathbb{Z}_p)$. Then $G = F_0 \coprod \mathbb{Z}_p^2 = F_0 \coprod (\langle y\rangle \times \mathbb{Z}_p)$ embeds in $F_0 \coprod (F_2 \coprod_{\langle y\rangle}(\langle y\rangle \times \mathbb{Z}_p))$.

We claim that there is a basis $\{x_1, \ldots, x_s\}$ of the free pro-p group $F = F_2 \coprod F_0$ such that

$$y = [x_1, x_2]t,$$

where t belongs to $\{x_3, \ldots, x_s\}$, and hence the image of x_2 in $F/\langle y\rangle^F$ is non-trivial. Indeed let x_1, x_2, y be a part of a basis of F. Then $x_1, x_2, t = [x_2, x_1]y$ is a subset of a basis $\{x_1, x_2, x_3 = t, \ldots\}$ of F.

Finally we can apply Theorem 4.1 to deduce that $(F_2 \coprod F_0) \coprod_{\langle y\rangle}(\langle y\rangle \times \mathbb{Z}_p)$ is residually free pro-p. Hence $F_0 \coprod (F_2 \coprod_{\langle y\rangle}(\langle y\rangle \times \mathbb{Z}_p))$ is residually free pro-p, so its closed subgroup G is residually free pro-p as well. □

7. Fully residually free pro-p groups

Proposition 7.1. *Suppose that G is a residually free pro-p group and F_2 is a free pro-p group of rank 2. Then the following conditions are equivalent*:

1. *G is fully residually free pro-p*;
2. *G does not have a closed subgroup isomorphic to $F_2 \times \mathbb{Z}_p$*;
3. *G is commutative transitive i.e. for every $g \in G \setminus \{1\}$ the centralizer $C_G(g)$ is abelian*;
4. *G is CSA i.e. every closed maximal abelian subgroup of G is malnormal.*

If G has trivial center the following condition can be added to the above list:

5. *G does not have a closed subgroup isomorphic to $F_2 \times F_2$.*

If G is non-abelian the following condition can be added to the above list:

6. *for every $g \in G \setminus \{1\}$ and N_g the normal closed subgroup of G generated by g the centralizer $C_G(N_g) = 1$.*

Proof. In the abstract case the equivalence between the different conditions stated above (with $\mathbb{Z}_p$ substituted with $\mathbb{Z}$) was proved in several of the results from [1] i.e. 1 is equivalent with 2 is [1, Theorem 3], 1 is equivalent with 3 is [1, Theorem 1], 1 is equivalent with 5 is [1, Theorem 4], 1 is equivalent with 6 is [1, Theorem 2] and 3 is equivalent with 4 follows immediately from the definition of malnormality.

It is worth mentioning that the proofs in [1] of the equivalence of the above conditions in the abstract case use general constructions like centralizers, subgroups and the result that two generated abstract non-abelian residually free group is free [2, Lemma 1] but do not use combinatorial properties of words in abstract groups. Note that [2, Lemma 1] holds for pro-p groups: for a non-abelian residually free pro-p group M generated by two elements, say x and y, there is some image F of M such that F is free pro-p and the image of $[x, y]$ in F is non-trivial. Then F is free of rank two and

$M \simeq F$ is free pro-p. Thus the same proofs as in [1] apply in the pro-p case substituting subgroups by closed subgroups and $\mathbb{Z}$ by $\mathbb{Z}_p$. $\square$

A Demushkin group D is a finitely generated pro-p group such that $H^2(D, \mathbb{F}_p) = \mathbb{F}_p$ and the product

$$\cup : H^1(D, \mathbb{F}_p) \times H^1(D, \mathbb{F}_p) \to H^2(D, \mathbb{F}_p) \simeq \mathbb{F}_p$$

is a non-singular bilinear product. Such a group D if infinite is a pro-p Poincare duality group of dimension 2 [10, 3.7.6].

Lemma 7.2. *Every infinite Demushkin group G is transitive commutative. In particular any orientable surface pro-p group of any genus is transitive commutative.*

Proof. Let $g \in G \setminus \{1\}$. We claim that $C_G(g)$ is abelian. Indeed if $C_G(g)$ has finite index in G then $C_G(g)$ is a Demushkin group with non-trivial center, hence $C_G(g) \simeq \mathbb{Z}_p^2$. If $C_G(g)$ has infinite index in G then $C_G(g)$ is free pro-p with non-trivial center, so $C_G(g) \simeq \mathbb{Z}_p$. $\square$

Theorem 7.3. *Every orientable surface pro-p group*

$$G_d = \langle x_1, \ldots, x_{2d} \mid [x_1, x_2] \ldots [x_{2d-1}, x_{2d}] = 1 \rangle,$$

is fully residually free pro-p provided d is even or $p = 2$.

Proof. By Lemma 7.2 G_d is transitive commutative. Then by Corollary 5.1, Corollary 5.2 and Theorem 7.1 G_d is fully residually free pro-p. $\square$

Theorem 7.4. *Let F_0 be a free pro-p group of finite rank. Then $G = F_0 \coprod \mathbb{Z}_p^2$ is fully residually free pro-p.*

Proof. By the proof of Theorem 6.1 the pro-p group G belongs to the class $\mathcal{L}$ defined in [6]. Then G is transitive commutative. Then by Theorem 6.1 and Theorem 7.1 G is fully residually free pro-p. $\square$

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